Classifying diff EQ's

Homogenious? $y' + y = 0$

Linear? no y^{power} no functions of y

power? = highest derivative

constant coeficient? = no ty

Picards theorem

Given an IVP

If $\frac{dy}{dt}$ or $f(y,t)$ is defined garuentees 1 solution

If $f_y(t,y)$ is defined garuentees a unique solution

If ND cant tell unique

must be cont.

Mixing

$$\frac{dy}{dt} = (\text{conc. in})(\text{flow in}) - (\text{conc. out})(\text{flow out})$$

Ex: 1 g/L, 6 L/min → 50 g, 360 L → 6 L/min

$\frac{6L}{m} \cdot \frac{1g}{L}$

$$\frac{dA}{dt} = 6 - \frac{6}{360}A$$

Integrating Factor

$$y' + p(t)y = f(t)$$

$$\mu = e^{\int p(t)\,dt}$$

$$\frac{1}{\mu}\left(\int f(t)\cdot\mu\,dt + C\right)$$

Newtons Cooling

$$T(t) = m + (T_0 - m)e^{-kt}$$

m = outside temp, T_0 = start temp

Isoclines

curves in ty plane where slope constant

replace y' with c (slope), graph

Nullclines

slope in ty plane where slope 0

v nullcline $\frac{dx}{dt} = 0$

h nullcline $\frac{dy}{dt} = 0$

Singular Matrix

det = 0 or A^{-1} not defined

Euler step Method

$ty' - 2y = t^4e^{t^2}$, $t > 0$

$y_h = e^{\int -2/t\,dt} = t^2C$ plug in

$y_p = vt^2$, $y_p' = v't^2 + 2tv$

$t(v't^2 + 2tv) - 2(vt^2) = t^4e^{t^2}$

$v't^3 + \cancel{2t^2v} - \cancel{2vt^2} = t^4e^{t^2}$

$\int v' = \int te^{t^2} \Rightarrow v = \frac{1}{2}e^{t^2}$

$y_p = \frac{1}{2}e^{t^2}t^2$ + $y_h = Ct^2$

$$\boxed{y = \tfrac{1}{2}e^{t^2}t^2 + Ct^2}$$

Exponential Growth + Decay

$y = Ce^{kt}$ growth

$y = Ce^{-kt}$ decay

for $y(0) = y_0$, $C = y_0$

Compound Interest

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

r = interest rate

n = # times per year compounded

Equilibrium Solutions

$\frac{dy}{dt} = 0$ and no t dependence

Autonomous? if can be written as $\frac{dy}{dt} = f(y)$

Phase line

stable, semi stable, unstable

equilibrium at intersection of V_{null} + h_{null}

Linear Independence?

1. det ≠ 0 of matrix made of collumn vectors
2. can be taken down to RREF

Rank(A) = # pivit collumns

nullity(A) = # vectors

investable?

- det ≠ 0
- RREF = I
- Rank = 4

Variation of paramaters

$v_1' = \frac{-y_2 f(x)}{W}$

$v_2' = \frac{y_1 f(x)}{W}$

$y_p = v_1y_1 + v_2y_2$

$y(x) = y_h + y_p$

Other matrix rules

$(A^{-1})^{-1} = A$

cannot add matrixs of different sizes

can only multiply if $[m \times n][n \times m]$

Matrix Algebra

$$|BB^TB^{-1}| = |B||B^T||B^{-1}| = |B||B|\left(\frac{1}{|B|}\right) = |B|$$

$AB \neq BA$

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$(B+C)A = BA + CA$

$(A^{-1})^{-1} = A$

$(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1}A^{-1}$

$|AB| = |A|\cdot|B|$

$|A+B| \neq |A| + |B|$

$|A^T| = |A|$

$|A^{-1}| = \frac{1}{|A|}$ ← must be non-zero

Singular Matrix

Ex: $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$b^2 < 4ak \Rightarrow$ 2 complex roots

underdamped

$\alpha = -\frac{b}{2m}$, $\beta = \frac{\sqrt{b^2 - 4ak}}{2m}$

$y_h = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$

$b^2 > 4mk \Rightarrow$ overdamped

2 real solutions

$y_h = c_1e^{r_1t} + c_2e^{r_2t}$

Springs

solutions grow without bound

$F_0 \neq 0$

$b = 0$ no damping

resonance $w_0 = w_f$

$F = kx$

pass through t axis more than once

↳ criticly damped or overdamped

$$w_0 = \sqrt{\frac{k}{m}}$$

pass thru equilibrium position

t = ? when x(t) = 0

Subspace Test

P·A + Q·B

zero vector must be in subspace

A and B elements of subspace

Let A and B be f(t) and g(t) and $\int_0^1 f(t)\,dt = 0$ and $\int_0^1 g(t)\,dt = 0$

P and Q are constants

$$\int_0^1 Pf(t)\,dt + \int_0^1 Qf(t)\,dt = 0$$

$P(0) + Q(0) = 0$ ✓

thus in subspace

unbounded: underdamped with resonance

grow without bounds: Forced with resonance

Pass Thru one: critically or underdamped

Forced Harmonic Oscilator

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(w_ft)$$

resonance: $b = 0$, $w_0 = w_f$, $F_0 \neq 0$

$w_0 = \sqrt{\frac{k}{m}}$

growth: $\frac{F_0}{2mw_0}t$

$x_p = \frac{F_0}{2mw_0}t\sin(w_0t)$

$x_h = \cdots$

$x(t) = x_p + x_h$

Conservative system

↳ $b = 0$ (friction), $f(t) = 0$ (forcing term)

$V(x) = \int F(x)\,dx$ ← potential

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x)$$

$\frac{1}{2}kx^2$

Form a Basis?

Linearly independent

correct dimension

$\mathcal{L}\{f^n\}$

$y'' = s^2Y(s) - sy(0) - y'(0)$

$y' = sY(s) - y(0)$

$y = Y(s)$

$$\mathcal{L}\{f(t)H(t-c)\} = e^{-sc}\mathcal{L}\{f(t-c)\}$$

$y^{(4)} + y'' = -\sin(t)$, $y(0) = 2$, $y'(0) = y''(0) = y'''(0) = 1$

want in form $\vec{x}' = A\vec{x} + \vec{f}$

$x_1 = y$, $x_1' = y' = x_2$

$x_2 = y'$, $x_2' = y'' = x_3$

$x_3 = y''$, $x_3' = y''' = x_4$

$x_4 = y'''$, $x_4' = y'''' = -\sin(t) - x_3$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 1 \\ x_3(0) = 1 \\ x_4(0) = 1 \end{bmatrix}$$

Transient vs Steady State

Transient whats left after $t \to \infty$

steady state whats oscilating

$$\tan\delta = \frac{bw_f}{m(w_0^2 - w_f^2)}$$

$$A(w_f) = \frac{F_0}{\sqrt{m^2(w_0^2 - w_f^2)^2 + (bw_f)^2}}$$

undetermined Coefitients

if trial has characteristic multiply by t

f(x)	trial
te^t	$(At + B)e^t$
$3t^2e^t$	$(At^2 + Bt + C)e^t$
$e^t\sin(2t)$	$e^t(A\cos 2t + B\sin 2t)$
$t\sin t$	$(At + B)\cos t + (Ct + D)\sin t$
$2t^{-1}$	not applicable
$e^t + te^t$	$Ae^t + (Bt + C)e^t$

Eulers Method

$$T_{n+1} = T_n + (\text{step size})f(T)$$

For repeated real Eigenvalues:

① Find eigenvector $\vec{v}$

② Find $\vec{u}$ such that $(A - \lambda I)\vec{u} = \vec{v}$

③ Solution is $\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})$

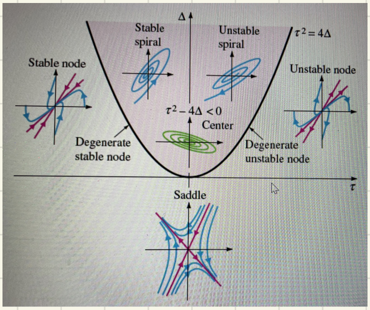
Imaginary Eigenvalues

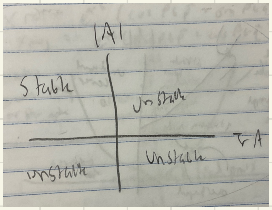
$$\underbrace{e^{\alpha t}(\cos\beta t \cdot \vec{p} - \sin\beta t\,\vec{q})}_{\text{real}} + \underbrace{e^{\alpha t}(\cos\beta t\,\vec{q} + \sin\beta t\,\vec{p})}_{\text{Imag}}$$

2 Real Eigenvalues

$$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

Classification





uniqe solution as long as $\det|A| \neq 0$

linear system non-trivial solutions if $\det|A| = 0$

↳ 0 eigenvalue

If Real distinct λ:

- $\lambda_1 < \lambda_2 < 0$ attracting node
- $0 < \lambda_1 < \lambda_2$ repelling node
- $\lambda_1 < 0 < \lambda_2$ saddel point

Complex λ: $\lambda = \alpha \pm i\beta$

- $\alpha < 0$ attracting spiral
- $\alpha > 0$ repelling spiral
- $\alpha = 0$ center → Neutrally stable

Find equilibrium points?

$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ → (x,y) where they both happen